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Note on the class of unit in
 $K_0(C(S^1) \times_{\Gamma} \Gamma)$
(Operator Algebras
and Applications)

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CITATION:

Natsume, Toshikazu. Note on the class of unit in $K_0(C(S^1) \times_{\Gamma} \Gamma)$
(Operator Algebras and Applications). 数理解析研究所講究録 1985, 560: 1-11

ISSUE DATE:

1985-05

URL:

<http://hdl.handle.net/2433/99033>

RIGHT:

Note on the class of unit in $K_0(C(S^1) \rtimes_r \Gamma_g)$

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1. Prologue:

In [2] A. Connes showed, among other things, that the unit of the reduced crossed product $C(S^1) \rtimes_r \Gamma$ is a torsion element of the K_0 -group for every torsionfree cocompact discrete subgroup Γ of $PSL_2(\mathbb{R})$ acting on S^1 , which we view as the boundary of the upper half-plane, given by the linear fractional transformations. He showed this in the framework of his noncommutative differential geometry. More precisely, he showed that for a suitable element x of the geometric group $K^0(S^1, \Gamma)$ which corresponds to the class of unit through the index map $\mu : K^0(S^1, \Gamma) \rightarrow K_0(C(S^1) \rtimes_r \Gamma)$, the Chern character $ch(x)$ is equal to zero. Consequently, x is a torsion element, hence so is the class of unit.

The purpose of this note is to give an elementary proof of this result in a purely C^* -algebraic manner.

This work was inspired during helpful conversations with G.A. Elliott while the author was enjoying the hospitality of the Mathematics Institute of the University of Copenhagen. He takes this opportunity to express his gratitude to Elliott and the Institute.

2. The group Γ_g :

Let Γ_g be the fundamental group of a closed Riemannian surface of genus $g > 1$. We can regard Γ_g as a torsion-free cocompact discrete subgroup of $\text{PSL}_2(\mathbb{R})$.

It is well-known that Γ_g is generated by $2g$ generators $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ with a single relation:

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1.$$

Put $\gamma = [\beta_1, \alpha_1] \ (\ = [\alpha_2, \beta_2] \cdots [\alpha_g, \beta_g])$. Let G (resp. S) be the group generated by α_1, β_1 (resp. $\alpha_2, \beta_2, \dots, \alpha_g, \beta_g$). Then G and S are free groups with 2 and $2g-2$ generators, respectively. From the above relation, it follows that

$$\Gamma_g = G \star_H S,$$

where H is the infinite cyclic subgroup generated by γ .

Let A, A_G, A_S and A_H denote the reduced crossed products $C(S^1) \rtimes_{\Gamma_g}, C(S^1) \rtimes_G, C(S^1) \rtimes_S$ and $C(S^1) \rtimes_H$, respectively. Let κ^1 (resp. κ^2) be a natural inclusion of A_H into A_G (resp. A_S), and let ε^1 (resp. ε^2) be a natural inclusion of A_G (resp. A_S) into A . Then

$$\varepsilon^1 \circ \kappa^1 = \varepsilon^2 \circ \kappa^2.$$

In the latter sections, we compute the induced maps κ_\star^1 and κ_\star^2 in K -theory.

3. Rieffel projection:

Let h be an orientation preserving homeomorphism of S^1 , which is not the identity.

Definition 1. A foundation of h is a quadruple $T = (t_0, t_1, t_2, t_3)$ of mutually distinct points of S^1 such that

- 1) $h(t_0) = t_2$, $h(t_1) = t_3$;
- 2) t_0, \dots, t_3 are contained in the same connected component I of $S^1 \setminus \text{Fix}(h)$ and are sitting in the negative direction in I , which we view as an oriented open submanifold of S^1 .

We give examples. Let h be induced from a matrix of the form

$$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

with $p > 0$. Then h has no foundations. On the other hand, if h is induced from a hyperbolic element, then h always has a foundation. Thus, an orientation preserving homeomorphism h may or may not have a foundation. However, at least one of h and h^{-1} has a foundation.

Let $T = (t_0, t_1, t_2, t_3)$ be a foundation of h . The homeomorphism h gives rise to an action of \mathbb{Z} on $C(S^1)$. Consequently, the crossed product $C(S^1) \rtimes \mathbb{Z}$ is defined.

For $i = 0, 1, 2$, let $[t_i, t_{i+1}]$ denote the closure of the connected component of $S^1 \setminus \{t_i, t_{i+1}\}$ which does not contain t_k , where $k \neq i, k \neq i+1$.

On $[t_0, t_1]$ let f be a continuous function with values in $[0, 1]$ and with $f(t_0) = 0$ and $f(t_1) = 1$. On $[t_2, t_3]$ define f by

$$f(t) = 1 - f(h^{-1}(t)),$$

while on $[t_1, t_2]$ let f have value 1. On the complement of

$[t_0, t_1] \cup [t_1, t_2] \cup [t_2, t_3]$, let f have value 0. Define g by

$$g(t) = (f(t)(1-f(t)))^{1/2}.$$

Put $e = (gU_h)^* + f + gU_h$, where U_h is the canonical unitary of $C(S^1) \rtimes \mathbb{Z}$ corresponding to h . Then, from the construction, it follows that e is a projection of $C(S^1) \rtimes \mathbb{Z}$. We call e a Rieffel projection associated to h .

Notice that as h is orientation preserving, the action of \mathbb{Z} on $K_1(C(S^1))$ is trivial.

Let δ_1 be the connecting homomorphism from $K_0(C(S^1) \rtimes \mathbb{Z})$ into $K_1(C(S^1))$ associated to a six-term exact sequence obtained in [6]. Let z be the canonical unitary of $C(S^1)$. Then by a direct computation (cf., [Appendix, 6]) we get:

Lemma 2. We have $\delta_1([e]) = [z]$ in $K_1(C(S^1))$.

By Lemma 2 together with the six-term exact sequence mentioned above, it follows that $K_0(C(S^1) \rtimes \mathbb{Z})$ is a free abelian group generated by $[1]$ and $[e]$. We denote the class of e by $\beta(h, T)$.

Let T, T' be foundations of h . Then:

Lemma 3. For some $m \in \mathbb{Z}$, we have

$$\beta(h, T) - \beta(h, T') = m[1].$$

The proof is easy and omitted here.

4. Computations:

In this section we compute the maps κ_*^1 and κ_*^2 .

First, notice that Γ_g has no elliptic elements, because it acts freely on the upper half plane. By the construction of the embedding of Γ_g into $PSL_2(\mathbb{R})$, we can easily see that the α_i 's

and β_j 's have their own foundations.

We compute $\kappa_*^1 : K_0(A_H) \longrightarrow K_0(A_G)$. Choose a foundation of γ to get a Rieffel projection e corresponding to γ . Since γ has a fixed point, there exists a normalized trace τ on A_H . Using the induced map $\tau_* : K_0(A_H) \longrightarrow \mathbb{R}$ together with Lemma 3, we can show that the class $[e]$ of e does not depend on the choice of a foundation.

Take foundations and construct Rieffel projections p_1 and q_1 corresponding to α_1 and β_1 , respectively. Consider the C^* -dynamical system $(C(S^1), \mathbb{Z}, \rho)$ determined by α_1 . Let B be the reduced crossed product of the system $(C(S^1), \mathbb{Z}, \rho)$. By the same argument as above, we can show that the class $[p_1]$ of p_1 in $K_0(B)$ is independent of the choice of a foundation of α_1 , because α_1 has a fixed point. Consequently, the class $[p_1]$ in $K_0(A_G)$ is also independent of the choice of a foundation. Similarly, the class of q_1 in $K_0(A_G)$ is uniquely determined.

As we have seen in the previous section, $K_0(A_H)$ is a free abelian group generated by $[1]$ and $[e]$. By using a six-term exact sequence for A_G (cf., [7]), we can see that $K_0(A_G)$ is a free abelian group with generators $[1]$, $[p_1]$ and $[q_1]$.

Obviously, $\kappa_*^1([1]) = [1]$. Put

$$\kappa_*^1([e]) = k[p_1] + m[q_1] + n[1].$$

Consider the six-term exact sequence mentioned above:

$$\begin{array}{ccccc} K_0(C(S^1)) & \longrightarrow & K_0(C(S^1) \rtimes_r G') & \longrightarrow & K_0(A_G) \\ \delta_0 \downarrow & & & & \delta_1 \downarrow \\ K_1(A_G) & \longleftarrow & K_1(C(S^1) \rtimes_r G') & \longleftarrow & K_1(C(S^1)), \end{array}$$

where G' is the infinite cyclic subgroup of G generated by α_1 . By a direct computation using the argument of [Appendix, 6], we have that

$$\delta_1(\kappa_*^1([e])) = 0.$$

This means that $m = 0$, because $\delta_1([q_1]) = [z]$.

If we exchange the roles of α_1 and β_1 , we can see that k is also equal to zero. Thus we obtain:

$$\kappa_*^1([e]) = k[1]$$

for some $k \in \mathbb{Z}$.

Let $\tilde{\alpha}_1, \tilde{\beta}_1$ and $\tilde{\gamma}$ be the lifts of α_1, β_1 and γ , respectively, so that $\tilde{\alpha}_1, \tilde{\beta}_1$ and $\tilde{\gamma}$ have fixed points. As $\gamma = [\beta_1, \alpha_1]$, we see that

$$\tilde{\gamma} = [\tilde{\beta}_1, \tilde{\alpha}_1]\tau_n,$$

where τ_n is the translation of \mathbb{R} by n , namely, $\tau_n(x) = x + n$ with $n \in \mathbb{Z}$.

Proposition 4. We have $k = n$.

Proof. Let $j \in \mathbb{N}$ be fixed. Consider the j -fold covering $\pi_j : S^1 \longrightarrow S^1$. The diffeomorphisms $\tilde{\gamma}, \tilde{\alpha}_1$ and $\tilde{\beta}_1$ descend to diffeomorphisms $\bar{\gamma}, \bar{\alpha}_1$ and $\bar{\beta}_1$ of the total space of π_j , respectively. Since $\tilde{\gamma} = [\tilde{\beta}_1, \tilde{\alpha}_1]\tau_n$, we have that

$$\bar{\gamma} = [\bar{\beta}_1, \bar{\alpha}_1]\tau(n/j),$$

where $\tau(n/j)$ is the rotation by the angle $(2\pi n)/j$.

Let D be the crossed product $C(S^1) \rtimes \mathbb{Z}/(j)$, where the action is given by the rotation $\tau(1/j)$ with angle $(2\pi)/j$.

It is easy to see that $K_0(D)$ is isomorphic to \mathbb{Z} , in particular the class $[1_D]$ of the unit 1_D is torsion-free.

Notice that the diffeomorphisms $\bar{\alpha}_1$ and $\bar{\beta}_1$ commute with $\tau(1/j)$. Let σ be the action of the free group F_2 with two generators, defined by $\bar{\alpha}_1$ and $\bar{\beta}_1$, and let E be the associated reduced crossed product. Using the six-term exact sequence for crossed products by free groups [7], we can see that the class of the unit 1_E of E is not a torsion element of $K_0(E)$.

Let $\Pi = \pi_j^* : C(S^1) \longrightarrow C(S^1)$ be the map induced from π_j . Put $\Pi(\alpha_1) = \bar{\alpha}_1$ and $\Pi(\beta_1) = \bar{\beta}_1$. Then Π extends to a homomorphism from the full crossed product $C(S^1) \rtimes G$ into E . Since G is K -amenable [3], Π induces a homomorphism

$$\Pi_* : K_*(A_G) \longrightarrow K_*(E),$$

and $\Pi_*(\kappa_*^1([e])) = k[1_E]$.

Taking a foundation of $\bar{\gamma}$ which covers that of γ , we construct a Rieffel projection e' corresponding to $\bar{\gamma}$. Then, by the construction, we get:

$$\Pi_*(\kappa_*^1([e])) = j[e'].$$

On the other hand, from the equality $\bar{\gamma} = [\bar{\beta}_1, \bar{\alpha}_1]\tau(n/j)$, it follows that in $K_0(E)$, we have

$$[e'] = n[e''] + i[1_E]$$

for some $i \in \mathbb{Z}$, where e'' is a Rieffel projection corresponding to $\tau(1/j)$. Therefore

$$k[1_E] = jn[e''] + ji[1_E].$$

It is not difficult to see that $j[e] = [1_D]$ in $K_0(D)$. Consequently, $j[e] = [1_E]$ in $K_0(E)$. Hence $(k-n)[1_E] = ji[1_E]$. Since $[1_E]$ is not a torsion element of $K_0(E)$, we see that $k-n=ji$. This means that $k-n$ is divisible by j . As j is arbitrary, $k-n=0$. Q.E.D.

Since $\gamma = [\alpha_2, \beta_2] \cdots [\alpha_g, \beta_g]$, there exists $m \in \mathbb{Z}$ such that

$$\tilde{\gamma} = [\tilde{\alpha}_2, \tilde{\beta}_2] \cdots [\tilde{\alpha}_g, \tilde{\beta}_g] \tau_m.$$

Then as in the proof of Proposition 4, we see that

$$\kappa_*^2([e]) = m[1].$$

We have now an equality

$$[\tilde{\alpha}_1, \tilde{\beta}_1][\tilde{\alpha}_2, \tilde{\beta}_2] \cdots [\tilde{\alpha}_g, \tilde{\beta}_g] = \tau_{(k-m)}.$$

Then, by [8], we have $k-m=2g-2$.

Recall that $\varepsilon^1 \cdot \kappa^1 = \varepsilon^2 \cdot \kappa^2$. Therefore,

$$\begin{aligned} k[1_A] &= (\varepsilon^1 \cdot \kappa^1)_*([e]) \\ &= (\varepsilon^2 \cdot \kappa^2)_*([e]) \\ &= m[1_A]. \end{aligned}$$

From this it follows

$$(k-m)[1_A] = (2g-2)[1_A] = 0.$$

Thus we have:

Theorem 5 ([Corollary 6.7, 1]). For the fundamental group

$\Gamma_g \subset \text{PSL}_2(\mathbb{R})$ of a closed Riemannian surface of genus $g \geq 2$, in the reduced crossed product $A = C(S^1) \rtimes_{\Gamma_g}$ the unit 1_A is a torsion element of $K_0(A)$.

5. Epilogue:

The computations carried in the previous sections are exactly what we omitted in [Epilogue, 5].

We briefly review the main result of [5]. Let G be an amalgamated product of countable groups G_1 and G_2 along G_0 , namely,

$$G = G_1 \star_{G_0} G_2.$$

Let (A, α, G) be a C^* -dynamical system. In [5] we showed that if one of (G_1, G_0) and (G_2, G_0) has property Λ , then there exists a six-term exact sequence:

$$\begin{array}{ccccccc} K_0(A \rtimes_r G_0) & \longrightarrow & K_0(A \rtimes_r G_1) \oplus K_0(A \rtimes_r G_2) & \longrightarrow & K_0(A \rtimes_r G) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_r G) & \longleftarrow & K_1(A \rtimes_r G_1) \oplus K_1(A \rtimes_r G_2) & \longleftarrow & K_1(A \rtimes_r G_0). \end{array}$$

Let $\Gamma_g = G \star_H S$ be as in the preceding sections. So far, it is not known whether (G, H) or (S, H) has property Λ .

We have the following complex of abelian groups:

$$\begin{array}{ccccc} \kappa_*^1 - \kappa_*^2 & & \epsilon_*^1 + \epsilon_*^2 & & \\ K_*(A_H) & \longrightarrow & K_*(A_G) \oplus K_*(A_S) & \longrightarrow & K_*(A_{\Gamma_g}). \end{array}$$

As we have seen in Section 4, the maps $\kappa_*^1 - \kappa_*^2 : K_0(A_H) \rightarrow K_0(A_G) \oplus K_0(A_S)$ is given by the following matrix:

$$t \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & \cdots & 0 \\ m & 0 & 0 & 2g-2-m & 0 & \cdots & 0 \end{pmatrix} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \oplus \mathbb{Z}^{2g-1}.$$

It is rather easier to describe $\kappa_*^1 - \kappa_*^2 : K_1(A_H) \rightarrow K_1(A_G) \oplus K_1(A_S)$.

This map is given by

$$t \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \oplus \mathbb{Z}^{2g-1}.$$

Now assume the existence of a six-term exact sequence given in [Thm. A1, 5], for the C^* -dynamical system $(C(S^1), \Gamma_g, \alpha)$. Then

$$K_0(C(S^1) \rtimes_{\Gamma_g}) \simeq \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/(2g-2), \text{ and}$$

$$K_1(C(S^1) \rtimes_{\Gamma_g}) \simeq \mathbb{Z}^{2g+1}.$$

On the other hand, as $C(S^1) \rtimes_{\Gamma_g}$ is stably isomorphic to the C^* -algebra of an Anosov foliation on the unit tangent bundle of M_g , we can directly compute $K_*(C(S^1) \rtimes_{\Gamma_g})$ by using the Thom isomorphism (cf., [1]). The result coincides with the one above.

To conclude this section, we give the following:

Problem. Show that (G, H) or (S, H) has property Λ .

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